1.0 Limits and Continuity

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1.1 Limits

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LIMITS

- Limits are used to explain changes that arise for particular function when the value of an independent variable approaches a certain value.

- The limit is studied to see how its existence affects the shape of the graph of its function.

- Example:

$$f(x) = \frac{x-1}{x^2-1}$$

We know that the function will not defined if $x = 1$, but we still need to know what will happen to the value of $f(x)$ as x moves along the x axis approaching $x = 1$.

Example

Find the $\lim_{x\to 1} \frac{x-1}{x^2-1}$

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

Table shows that when x approaches 1, $f(x)$ approaches 0.5 and this represent the limit of the function.

$$\lim_{x\to 1^-} \frac{x-1}{x^2-1} = 0.5$$

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

Table shows that when x approaches 1, $f(x)$ approaches 0.5 and this represent the limit of the function.

$$\lim_{x\to 1^+} \frac{x-1}{x^2-1} = 0.5$$

"+" sign indicates that x approaches 1 from the right side

"-" sign indicates that x approaches 1 from the left side

These expressions also can called as **one-sided limits of $f(x)$**

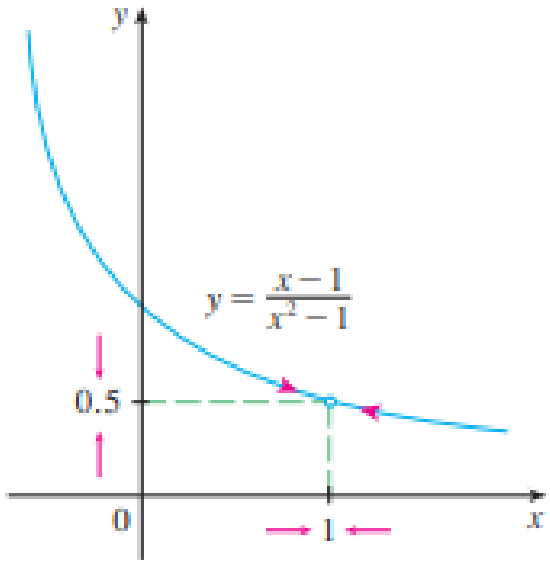
Example (continue..)

Since the limits of f(x) from the right side and the left side are the same and equal to 0.5, then we can write

$$\lim_{x\to 1^+} \frac{x-1}{x^2-1} = \lim_{x\to 1^-} \frac{x-1}{x^2-1} = 0.5$$

we can summarize the result as

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = 0.5$$



Limit of a Function

- If the function $f(x)$ approaches to a number L when x approaches x_0, then we say that the limit of $f(x)$ when x approaches x_0 is L.

$$\lim_{x \to x_0} f(x) = L$$

- If the limits from the left and the right sides of f(x) have the same value that is, if

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$$

- Then $\lim_{x \to x_0} f(x)$ **exist** and written as

$$\lim_{x \to x_0} f(x) = L$$

"limit of $f(x)$ as x approaches x_0 equals L".

Continue..

However if the limit from the left or right are not equal

$$\lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x)$$

Then $\lim_{x \to x_0} f(x)$ **does not exist.**

Example

Find the $\lim_{x \to 2} x^2 - x + 2$

x	$f(x)$
3.0	8.000000
2.5	5.750000
2.2	4.640000
2.1	4.310000
2.05	4.152500
2.01	4.030100
2.005	4.015025
2.001	4.003001

Table shows that when x approaches 2, $f(x)$ approaches 4 and this represent the limit of the function.

$$\lim_{x \to 2^?} x^2 - x + 2$$

x	$f(x)$
1.0	2.000000
1.5	2.750000
1.8	3.440000
1.9	3.710000
1.95	3.852500
1.99	3.970100
1.995	3.985025
1.999	3.997001

Table shows that when x approaches 2, $f(x)$ approaches 4 and this represent the limit of the function.

$$\lim_{x \to 2^?} x^2 - x + 2$$

Example

Find the $\lim_{x \to 2} x^2 - x + 2$

x	$f(x)$
3.0	8.000000
2.5	5.750000
2.2	4.640000
2.1	4.310000
2.05	4.152500
2.01	4.030100
2.005	4.015025
2.001	4.003001

Table shows that when x approaches 2, $f(x)$ approaches 4 and this represent the limit of the function.

$$\lim_{x \to 2^+} x^2 - x + 2$$

x	$f(x)$
1.0	2.000000
1.5	2.750000
1.8	3.440000
1.9	3.710000
1.95	3.852500
1.99	3.970100
1.995	3.985025
1.999	3.997001

Table shows that when x approaches 2, $f(x)$ approaches 4 and this represent the limit of the function.

$$\lim_{x \to 2^-} x^2 - x + 2$$

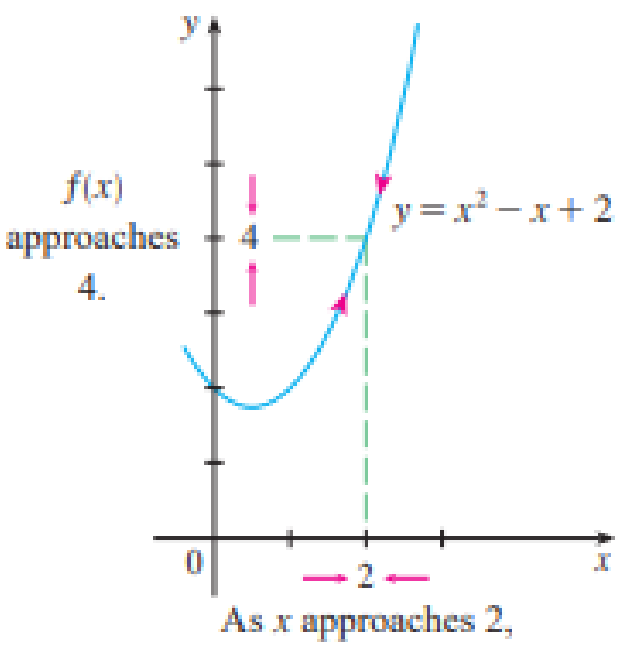
Example (continue..)

Since the limits of f(x) from the right side and the left side are the same and equal to 4, then we can write

$$\lim_{x \to 2^+} x^2 - x + 2 = \lim_{x \to 2^-} x^2 - x + 2 = 4$$

we can summarize the result as

$$\lim_{x \to 2} x^2 - x + 2 = 4$$



Sketch Graph

Sketch the graph of

$$f(x) = (x-1)^2,$$

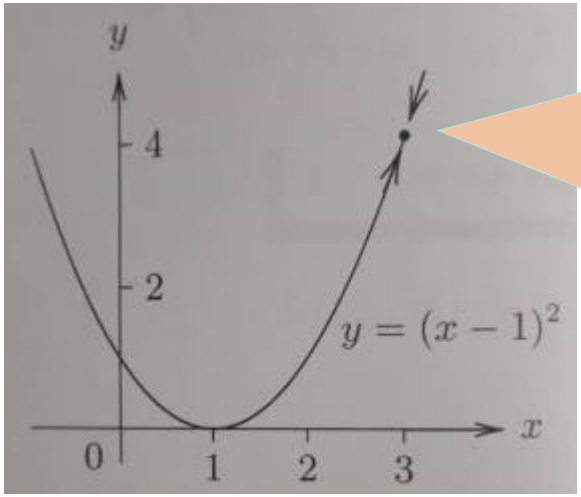
and evaluate $\lim_{x\to 3}(x-1)^2$

From the table as x approaching 3 from left and right sides.

x	4.0	3.5	3.1	3.05	3.01	3.005	3.001
F(x)	9.0000	6.2500	4.4100	4.2025	4.0401	4.0200	4.0040
x	2.1	2.5	2.8	2.9	2.95	2.995	2.999
F(x)	1.21	2.25	3.24	3.61	3.8025	3.9800	3.9960

Since limits from left and right have the same value,

$$\lim_{x\to 3}(x-1)^2 = 4$$



Imagine that the arrows in the figure represent two lovers running towards each other along the parabola.

It is 4, the limit value.

Sketch Graph

Sketch the graph of

$$f(x) = \frac{2x+1}{x-2},$$

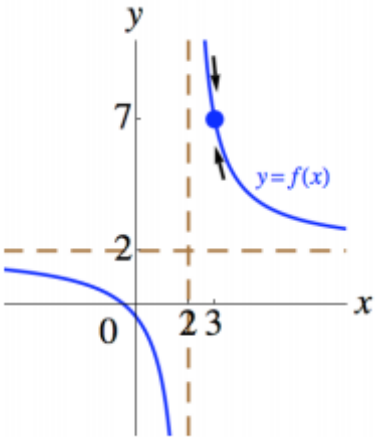
and evaluate $\lim_{x\to 3} f(x)$

From the table as x approaching 3 from left and right sides.

x	4.0	3.5	3.1	3.05	3.01	3.005	3.001
F(x)	4.5	5.33	6.545	6.762	6.95	6.9751	6.995
x	2.1	2.5	2.8	2.9	2.95	2.995	2.999
F(x)	52	12	8.25	7.555	7.263	7.025	7.005

Since limits from left and right have the same value,

$$\lim_{x\to 3} \frac{2x+1}{x-2} = 7$$



Sketch Graph

Sketch the graph of

$$f(x) = \begin{cases} x+1, & x < 1 \\ \frac{3}{2}, & x = 1 \\ 2-x, & x > 1 \end{cases}$$

and evaluate the limit of the function when $x \to 1$, if it exists.

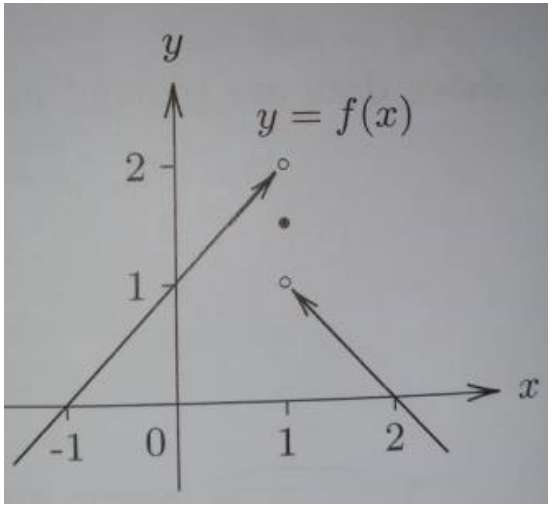
From the table as x approaching 1 from left and right sides.

x	-1	-0.5	-0.9	0.5	0.9	0.995	0.999
$x+1$	0	0.5	0.1	1.5	1.9	1.995	1.999

x	2.0	1.9	1.5	1.3	1.1	1.099	1.009
$2-x$	0	0.1	0.5	0.7	0.9	0.901	0.991

continue..

The graph of the function is given at the right



From the table as x approaches 1 from the left,

$$\lim_{x \to 1^-} f(x) = 2$$

From the table as x approaches 1 from the right,

$$\lim_{x \to 1^+} f(x) = 1$$

As the limit from right and left sides have different values,
$\therefore \lim_{x \to 1} f(x)$ does not exist.

$f(1) = \frac{3}{2}$ which is the value of the function at the point x=1. This value different from the value of the limits from left and right sides.

Practise 1.1

1. Evaluate and determine whether the limit f(x) exists.

a) $f(x) = 3x^2 - 1$ as $x \to 4$

b) $f(x) = \begin{cases} x - 1, x < 2 \\ x + 2, x \geq 2 \end{cases}$

c) $f(x) = \frac{2x-4}{x+5}$ as $x \to 3$

Rules of Limits

BASIC RULES OF LIMITS

Suppose a,c and n are real numbers. The limit of each of these expression is as follows

$$(a)\lim_{x\to a} c = c$$

$$(b)\lim_{x\to a} x = a$$

$$(c)\lim_{x\to a} x^n = a^n$$

EXAMPLE

$$(a)\lim_{x\to 0} 7 = 7$$

$$(b)\lim_{x\to \pi} x = \pi$$

$$(c)\lim_{x\to -2} x^3 = \left(-2\right)^3 = -8$$

Rules of Limits

Let $f(x)$ and $g(x)$ be two functions. If the limits of $f(x)$ and $g(x)$ exist at a particular point, then

$$(a)\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) \text{ for any constant } c$$

$$(b)\lim_{x\to a}\left[f(x)\pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$(c)\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$$

$$(\mathrm{d})\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{ where } \lim_{x\to a} g(x) \neq 0$$

$$(e)\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$$

$$(f)\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} \text{ provided } \lim_{x\to a} f(x) \geq 0 \text{ if } n \text{ is even.}$$

EXAMPLE

Using the rule of limits, evaluate the following limits

$$(a)\lim_{x\to\frac{1}{2}}\left(2x+7\right) \qquad (c)\lim_{x\to\pi}\frac{x^2-4}{x+2}$$

$$(b)\lim_{x\to 2}\left[5x^2\left(x-4\right)\right] \qquad (d)\lim_{x\to -2}\sqrt{5x^2-4}$$

SOLUTION:

$$(a)\lim_{x\to\frac{1}{2}}\left(2x+7\right)=2\left(\frac{1}{2}\right)+7=8$$

$$(b)\lim_{x\to 2}\left[5x^2\left(x-4\right)\right]=5\left(2\right)^2\left(2-4\right)=-40$$

$$(c)\lim_{x\to\pi}\frac{x^2-4}{x+2}=\frac{\left(\pi\right)^2-4}{\pi+2}=\frac{\left(\pi-2\right)\left(\pi+2\right)}{\left(\pi+2\right)}=\pi-2$$

$$(d)\lim_{x\to -2}\sqrt{5x^2-4}=\sqrt{5\left(-2\right)^2-4}=\sqrt{16}=4$$

Practice 1.2

1.Evaluate the following limits

$$(a)\lim_{x\to 1}\left(x^2-5\right)$$

$$(b)\lim_{x\to -2}\left(x^3-8x^2+4x-10\right)$$

$$(c)\lim_{x\to -1}\frac{2x^2-1}{x-3}$$

$$(d)\lim_{x\to 9}\frac{4\sqrt{x}+1}{1-\sqrt{x}}$$

2. For each of the following, state the value/s of k for the limit does not exist.

$$(a)\lim_{x\to k}\frac{5x+1}{\sqrt{x-4}}$$

$$(b)\lim_{x\to k}\frac{2}{\sqrt{x^2-4}}$$

Limits of Rational Functions

- Sometimes rational functions can be undefined when we apply the rules of limits.
- Therefore we need to simplify the rational function by **factorizing** or by **rationalizing** (multiply by conjugate)

- **EXAMPLE**
  Evaluate the following limit by using factorization

$$(a)\lim_{x\to 6}\frac{x^2-36}{x-6}$$

$$(b)\lim_{x\to 0}\frac{2x^2+5x}{x}$$

$$(c)\lim_{x\to -1}\frac{x^2+x}{x^2-1}$$

SOLUTION

$$(a)\lim_{x\to 6}\frac{x^2-36}{x-6}=\lim_{x\to 6}\frac{(x-6)(x+6)}{x-6}$$

$$=\lim_{x\to 6}(x+6)=12$$

$$(b)\lim_{x\to 0}\frac{2x^2+5x}{x}=\lim_{x\to 0}\frac{x(2x+5)}{x}$$

$$=\lim_{x\to 0}(2x+5)=5$$

$$(c)\lim_{x\to -1}\frac{x^2+x}{x^2-1}=\lim_{x\to -1}\frac{x(x+1)}{(x-1)(x+1)}$$

$$=\lim_{x\to -1}\frac{x}{(x-1)}=\frac{1}{2}$$

EXAMPLE

Evaluate the following limit by using rationalization

$$(a)\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}$$

$$(b)\lim_{x\to -1}\frac{x+1}{\sqrt{3-x}-2}$$

$$(c)\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9}$$

SOLUTION

$$(a)\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{(x-4)\left(\sqrt{x}+2\right)}{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}$$

$$=\lim_{x\to 4}\frac{(x-4)\left(\sqrt{x}+2\right)}{(x-4)}=\lim_{x\to 4}\left(\sqrt{x}+2\right)$$

$$=4$$

$$(b)\lim_{x\to -1}\frac{x+1}{\sqrt{3-x}-2}=\lim_{x\to -1}\frac{(x+1)\left(\sqrt{3-x}+2\right)}{\left(\sqrt{3-x}-2\right)\left(\sqrt{3-x}+2\right)}$$

$$=\lim_{x\to -1}\frac{(x+1)\left(\sqrt{3-x}+2\right)}{-(x+1)}=\lim_{x\to -1}-\left(\sqrt{3-x}+2\right)$$

$$=-4$$

SOLUTION

$$(c)\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9}=\lim_{x\to 9}\frac{\left(3-\sqrt{x}\right)\left(-3-\sqrt{x}\right)}{\left(x-9\right)\left(-3-\sqrt{x}\right)}$$

$$=\lim_{x\to 9}\frac{\left(x-9\right)}{\left(x-9\right)\left(-3-\sqrt{x}\right)}=\lim_{x\to 9}\frac{1}{\left(-3-\sqrt{x}\right)}$$

$$=-\frac{1}{6}$$

Practice 1.3

Evaluate the following limits using the suitable techniques.

$$(a)\lim_{x\to 7}\frac{x^2-49}{x+7}$$

$$(b)\lim_{x\to -1}\frac{x^2+6x+5}{x^2+3x+2}$$

$$(c)\lim_{x\to 2}\frac{\sqrt{x+7}-3}{x-2}$$

$$(d)\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x}$$

Cases of non-existence of limits

- As we have learned before this where the limit from the left and right sides have different values, therefore $\lim_{x \to x_0} f(x)$ does not exist.

- In other cases, $\lim\limits_{x \to x_0^+} f(x)$, $\lim\limits_{x \to x_0^-} f(x)$ or $\lim\limits_{x \to x_0} f(x)$ can not be determined such as $\lim\limits_{x \to 0} \frac{1}{x}$, $\lim\limits_{x \to 0} \frac{1}{x^2}$ where we cannot find the limit.

- In that situation , it called as **limit does not exist.**

Example

$$f(x)=\begin{cases} x-1, x<2 \\ x+2, x\geq 2 \end{cases}$$

Sketch the graph of $y=f(x)$, then determine whether $\lim\limits_{x\to 2} f(x)$ exists.

Solution:

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x-1)=2-1=1$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^-}(x+2)=2+2=4$$

Since $\lim\limits_{x\to 2^-} f(x)\neq \lim\limits_{x\to 2^+} f(x)$ then $\lim\limits_{x\to 2} f(x)$ does not exist.

Example

Let $f(x) = \frac{1}{x}$. Evaluate $\lim_{x \to 0} f(x)$

$\lim_{x \to 0^+} f(x)$

x	1.0	0.8	0.5	0.55	0.3	0.111	0.009
$1/x$	1	1.25	2	1.8181	3.33	9.000	111.11

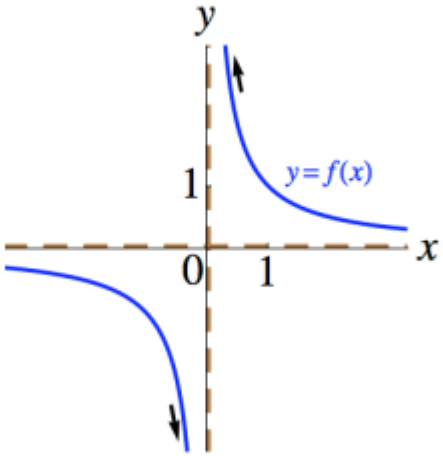
$\lim_{x \to 0^-} f(x)$

x	-1	-0.8	-0.5	-0.55	-0.3	-0.111	-0.009
$1/x$	-1	-1.25	-2	-1.818	-3.33	-9.000	-111.11

As x approaches 0 from the left or the right sides, the value of $f(x)$ increase indefinitely. We can write as

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \text{ and } \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$\therefore \lim_{x \to 0} \frac{1}{x}$ **does not exist**



Infinite Limits

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Introduction

- In the previous example, we find that for some functions, when x approaches a for both left and right, the value of $f(x)$ increases or decreases indefinitely.
- In such cases the function is said to have an infinite limit at $x = a$.

$i)\ \lim_{x \to a^-} f(x) = +\infty$. As x approaches a from left , $f(x)$ increases without bound.

$ii)\ \lim_{x \to a^-} f(x) = -\infty$. As x approaches a from left , $f(x)$ decreases without bound.

$iii)\ \lim_{x \to a^+} f(x) = +\infty$. As x approaches a from right , $f(x)$ increases without bound.

$iv)\ \lim_{x \to a^+} f(x) = -\infty$. As x approaches a from right , $f(x)$ decreases without bound.

(1) When $f(x)$ has an infinite limit at $x=a$, then the straight line $x=a$ is called a vertical limit.

If $\lim\limits_{x\to a^-} f(x)=L, L\neq 0$ and $\lim\limits_{x\to a^-} g(x)=0^+, \left(g(x)\right)$ approaches 0 positively, then

$$\lim_{x\to a^-}\frac{f(x)}{g(x)}=\begin{cases}+\infty, & \text{if } L>0\\ -\infty, & \text{if } L<0\end{cases}$$

(2) When $f(x)$ has an infinite limit at $x=a$, then the straight line $x=a$ is called a vertical limit.

If $\lim\limits_{x\to a^-} f(x)=L, L\neq 0$ and $\lim\limits_{x\to a^-} g(x)=0^-, \left(g(x)\right)$ approaches 0 negatively, then

$$\lim_{x\to a^-}\frac{f(x)}{g(x)}=\begin{cases}-\infty, & \text{if } L>0\\ +\infty, & \text{if } L<0\end{cases}$$

Example

Given $h(x)=\dfrac{x^2+1}{x+3}$. Find $\lim\limits_{x\to -3^-} h(x)$, $\lim\limits_{x\to -3^+} h(x)$ and $\lim\limits_{x\to -3} h(x)$.

Solution:

$h(x)=\dfrac{f(x)}{h(x)}$ so that $f(x)=x^2+1$ and $g(x)=x+3$

$$\lim_{x\to -3^-} f(x)=\lim_{x\to -3^-} x^2+1=(-3)^2+1=10$$

$$\lim_{x\to -3^-} g(x)=\lim_{x\to -3^-} x+3=?$$

x	**-3.1**	**-3.01**	**-3.001**
g(x)	-0.1	-0.01	-0.001

So $\lim\limits_{x\to -3^-} x+3=0^-$. Hence $\lim\limits_{x\to -3^-} \dfrac{x^2+1}{x+3}=-\infty$

Continue

Solution:

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} x^2+1 = (-3)^2+1 = 10$$

$$\lim_{x\to -3^+} g(x) = \lim_{x\to -3^+} x+3 = ?$$

x	**-2.999**	**-2.99**	**-2.9**
g(x)	0.001	0.01	0.1

So $\lim_{x\to -3^+} x+3 = 0^+$. Hence $\lim_{x\to -3^+} \dfrac{x^2+1}{x+3} = +\infty$

Therefore $\lim_{x\to -3} \dfrac{x^2+1}{x+3}$ does not exist.

Example

Let $f(x) = \frac{2}{x^2}$. Evaluate $\lim_{x \to 0} f(x)$

$\lim_{x \to 0^+} x^2$

x	0.1	0.01	0.001
x^2	0.01	0.0001	0.000001

$\lim_{x \to 0^+} x^2 = 0^+$

$\lim_{x \to 0^-} x^2$

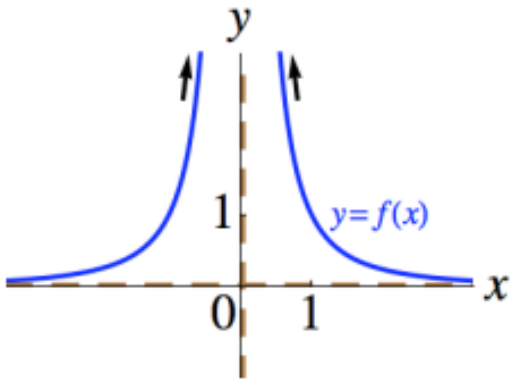
x	-0.1	-0.01	-0.001
x^2	0.01	0.0001	0.000001

$\lim_{x \to 0^-} x^2 = 0^+$

As x approaches 0 from the left or the right sides, the value of $f(x)$ increase indefinitely. We can write as

$$\lim_{x \to 0^+} \frac{2}{x^2} = +\infty \text{ and } \lim_{x \to 0^-} \frac{2}{x^2} = +\infty$$

$$\therefore \lim_{x \to 0} \frac{2}{x^2} = +\infty$$



where the notation "$+\infty$" signifies **the limit does not exist**

Practice 1.4

$$(a)\ \lim_{x\to 4^+} \frac{3}{x-4}$$

$$(b)\ \lim_{x\to -0.5^-} \frac{1+x}{2x+1}$$

$$(c)\ \lim_{x\to 1-} \frac{3x}{\sqrt{1-x^2}}$$

$$(d)\ \lim_{x\to 3} \frac{-3}{(x-3)^2}$$

Limits at Infinity

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Objectives

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

37

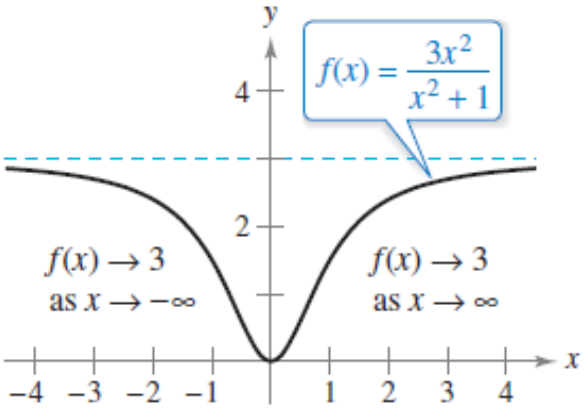
Limits at Infinity.

- We will study the behaviour of a function as the independent variable increase or decrease indefinitely.

- Increase indefinitely : $\lim_{x \to +\infty} f(x) = L$
- Decrease indefinitely : $\lim\limits_{x \to -\infty} f(x) = L$

Limits at Infinity

This section discusses the “end behavior” of a function on an *infinite* interval. Consider the graph of $f(x) = \dfrac{3x^2}{x^2 + 1}$ as shown in Figure 3.32.



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.

Figure 3.32

39

Limits at Infinity

Graphically, you can see that the values of $f(x)$ appear to approach 3 as x increases without bound or decreases without bound. You can come to the same conclusions numerically, as shown in the table.

x decreases without bound. | x increases without bound.

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\rightarrow \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.9703	1.5	0	1.5	2.9703	2.9997	$\rightarrow 3$

$f(x)$ approaches 3. | $f(x)$ approaches 3.

40

Limits at Infinity

The table suggests that the value of *f*(*x*) approaches 3 as *x* increases without bound $(x \to \infty)$. Similarly, *f*(*x*) approaches 3 as *x* decreases without bound $(x \to -\infty)$.

These **limits at infinity** are denoted by

$$\lim_{x \to -\infty} f(x) = 3$$ Limit at negative infinity

and

$$\lim_{x \to \infty} f(x) = 3.$$ Limit at positive infinity

41

Limits at Infinity

When evaluating limits an infinity, the next theorem is helpful.

Theorem 1 **Limits at Infinity**

If r is a positive rational number and c is any real number, then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \to -\infty} \frac{c}{x^r} = 0.$$

42

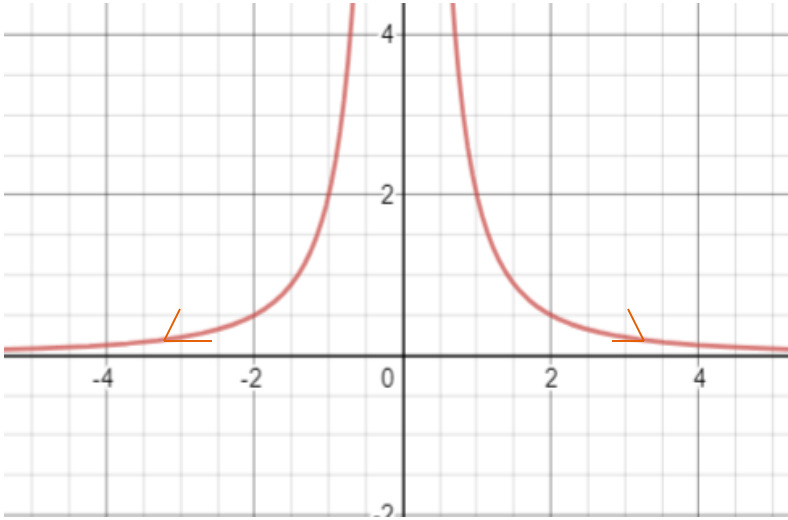
Example

Sketch the graph $y = \frac{2}{x^2}$, then evaluate $\lim_{x\to+\infty} y$ and $\lim\limits_{x\to-\infty} y$.

From the above theorem, we can write that

$\lim\limits_{x\to+\infty} \frac{2}{x^2} = 0$ and similarly, $\lim\limits_{x\to-\infty} \frac{2}{x^2} = 0$

From the graph itself, note that as x approaches $+\infty$, the value of y tends to 0, same goes to x approaches $-\infty$, value of y tends to 0.



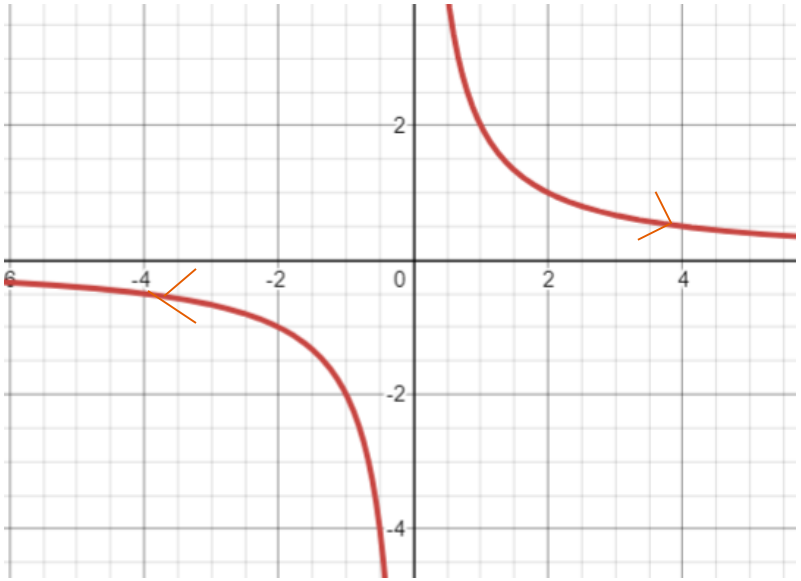
Example

Sketch the graph $y = \frac{2}{x}$, then evaluate $\lim_{x\to+\infty} y$ and $\lim\limits_{x\to-\infty} y$.

From the above theorem, we can write that

$\lim\limits_{x\to+\infty} \frac{2}{x} = 0$ and similarly, $\lim\limits_{x\to-\infty} \frac{2}{x} = 0$

From the graph itself, note that as x approaches $+\infty$, the value of y tends to 0, same goes to x approaches $-\infty$, value of y tends to 0.



Example 1 – *Finding a Limit at Infinity*

Find the limit: $\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right).$

Solution:

Using theorem 1, you can write

$$\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right)=\lim_{x\to\infty}5-\lim_{x\to\infty}\frac{2}{x^2}$$ Property of limits

$$=5-0$$

$$=5.$$

45

Limits at Infinity

- Above example x approaches to a finite value x_0. Now we will evaluate the limits when x approaches positive or negative infinity.

- Steps:

1. Divide the numerator and denominator of $f(x)$ with x^n, where n is the highest power of x in the denominator's term.
2. Apply the limits theorem.

Example – *Finding a Limit at Infinity*

Find the limit: $\lim_{x \to \infty} \frac{2x - 1}{x + 1}$.

Solution:

Note that both the numerator and the denominator approach infinity as *x* approaches infinity.

$$\lim_{x \to \infty} \frac{2x - 1}{x + 1} \to \lim_{x \to \infty} (2x - 1) \to \infty$$

$$\lim_{x \to \infty} \frac{2x - 1}{x + 1} \to \lim_{x \to \infty} (x + 1) \to \infty$$

47

Example – *Solution*

cont'd

This results in $\frac{\infty}{\infty}$, an **indeterminate form.** To resolve this problem, you can divide both the numerator and the denominator by *x*. After dividing, the limit may be evaluated as shown.

$$\lim_{x\to\infty} \frac{2x-1}{x+1} = \lim_{x\to\infty} \frac{\dfrac{2x-1}{x}}{\dfrac{x+1}{x}}$$

Divide numerator and denominator by x.

$$= \lim_{x\to\infty} \frac{2-\dfrac{1}{x}}{1+\dfrac{1}{x}}$$

Simplify.

$$= \frac{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \dfrac{1}{x}}{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \dfrac{1}{x}}$$

Take limits of numerator and denominator.

48

Example – *Solution*

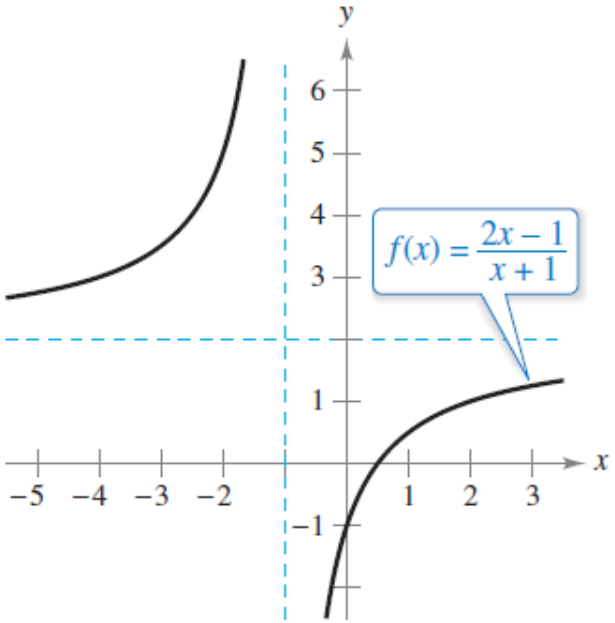
cont'd

$$= \frac{2 - 0}{1 + 0}$$ Apply Theorem 3.10.

$$= 2$$

So, the line $y = 2$ is a horizontal asymptote to the right.
By taking the limit as $x \to -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left.

The graph of the function is shown in Figure 3.35.



$y = 2$ is a horizontal asymptote.

Figure 3.35

49

Example – *Finding a Limit at Infinity*

Find the limit: $\lim_{x\to\infty}\frac{1+2x^3}{3-2x^3}$

Solution:

The highest power of x in denominator is x^3.

$$\lim_{x\to\infty}\frac{1+2x^3}{3-2x^3}=\lim_{x\to\infty}\frac{\frac{1+2x^3}{x^3}}{\frac{3-2x^3}{x^3}}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x^3}+2}{\frac{3}{x^3}-2}=\frac{\lim_{x\to\infty}\frac{1}{x^3}+2}{\lim_{x\to\infty}\frac{3}{x^3}-2}$$

$$=\frac{0+2}{0-2}=-1$$

Example

Find the limit: $\lim_{x\to-\infty} \frac{x-4x^2}{5-6x^3}$

Solution:

$$\lim_{x\to-\infty} \frac{x-4x^2}{5-6x^3}, \quad (\textit{the highest power of } x \textit{ in denominator is } x^3)$$

$$= \lim_{x\to-\infty} \frac{\frac{x}{x^3}-\frac{4}{x^3}}{\frac{5}{x^3}-\frac{6x^3}{x^3}} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}-\frac{4}{x}}{\frac{5}{x^3}-6}$$

$$= \frac{0-0}{0-6} = 0$$

Example

Find each limit.

a. $\lim_{x \to \infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$

b. $\lim_{x \to -\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}}$

Example (a)– *Solution*

For $x > 0, (+\infty)$ you can write $x = \sqrt{x^2}$.

So, dividing both the numerator and the denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\dfrac{3x-2}{x}}{\dfrac{\sqrt{2x^2+1}}{\sqrt{x^2}}} = \frac{3-\dfrac{2}{x}}{\sqrt{\dfrac{2x^2+1}{x^2}}} = \frac{3-\dfrac{2}{x}}{\sqrt{2+\dfrac{1}{x^2}}}$$

and you can take the limits as follows.

$$\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{3-\dfrac{2}{x}}{\sqrt{2+\dfrac{1}{x^2}}} = \frac{3-0}{\sqrt{2+0}} = \frac{3}{\sqrt{2}}$$

53

Example (b) – *Solution*

cont'd

For $x < 0$, $(-\infty)$ you can write $x = -\sqrt{x^2}.$

So, dividing both the numerator and the denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\dfrac{3x-2}{x}}{\dfrac{\sqrt{2x^2+1}}{-\sqrt{x^2}}} = \frac{3-\dfrac{2}{x}}{-\sqrt{\dfrac{2x^2+1}{x^2}}} = \frac{3-\dfrac{2}{x}}{-\sqrt{2+\dfrac{1}{x^2}}}$$

and you can take the limits as follows.

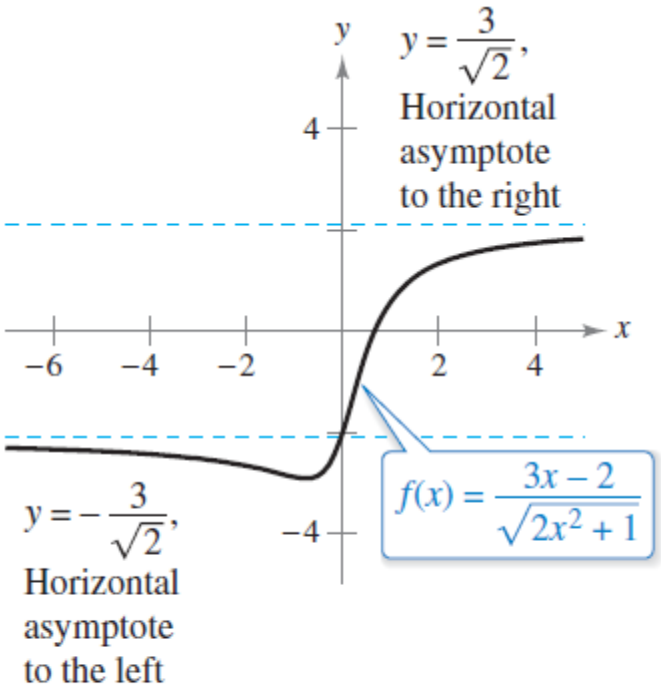
$$\lim_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to-\infty}\frac{3-\dfrac{2}{x}}{-\sqrt{2+\dfrac{1}{x^2}}} = \frac{3-0}{-\sqrt{2+0}} = -\frac{3}{\sqrt{2}}$$

54

Example – *Solution*

cont'd

The graph of $f(x) = (3x - 2)/\sqrt{2x^2 + 1}$ is shown in Figure 3.38.



Functions that are not rational may have different right and left horizontal asymptotes.

Figure 3.38

55

Practise 1.5

1. Find $\lim_{x\to\infty} f(x)$ for each of the following functions.

a) $f(x) = \frac{2x-4}{x+5}$

b) $f(x) = \frac{-4}{x^2+3}$

$c)\ \frac{x^2-1}{x+2}$

$d)\ \frac{2x^5+99}{x^6-1}$

2. Find

a) $\lim_{x\to+\infty} \frac{\sqrt{x^2+9}}{x+4}$

b) $\lim_{x\to-\infty} \frac{\sqrt{x^2+16}}{x-3}$

$c)\ \frac{2x+3}{\sqrt{9x^2-3x+2}}$

$d)\ \frac{-2}{\sqrt{x^2-3x+1}}$

Asymptotes

In the previous figure, Figure 3.35, the graph of *f* approaches the line $y = L$ as x increases without bound. The line $y = L$ is called a **horizontal asymptote** of the graph of *f.*

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L.$$

In sketching the graph of rational function, **vertical asymptotes** are associated with **infinite limits** whereas **horizontal asymptotes** are associated with **limits at infinity**.

Horizontal Asymptotes

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Example

Given $f(x) = \dfrac{x}{4 - x^2}$

Find the axes of intercepts, horizontal asymptotes and vertical asymptotes. Hence sketch the graph.

Solution:

STEP 1: For axes intercepts, let $y=0$, then $x=0$. Therefore the curve passing through (0,0).

STEP 2: Determine the vertical asymptotes.

Set the denominator $4 - x^2 = 0$. Hence $x = \pm 2$.

For $x=-2$ $\lim\limits_{x \to -2^-} \dfrac{x}{4 - x^2} = +\infty$ and $\lim\limits_{x \to -2^+} \dfrac{x}{4 - x^2} = -\infty$

continue..

CONTINUE STEP 2:

For x=2 $\lim_{x\to 2^-}\frac{x}{4-x^2}=+\infty$ and $\lim_{x\to 2^+}\frac{x}{4-x^2}=-\infty$

Therefore *x* = *2* and *x* = -2 is a vertical asymptote.

STEP 3: Determine horizontal asymptote.

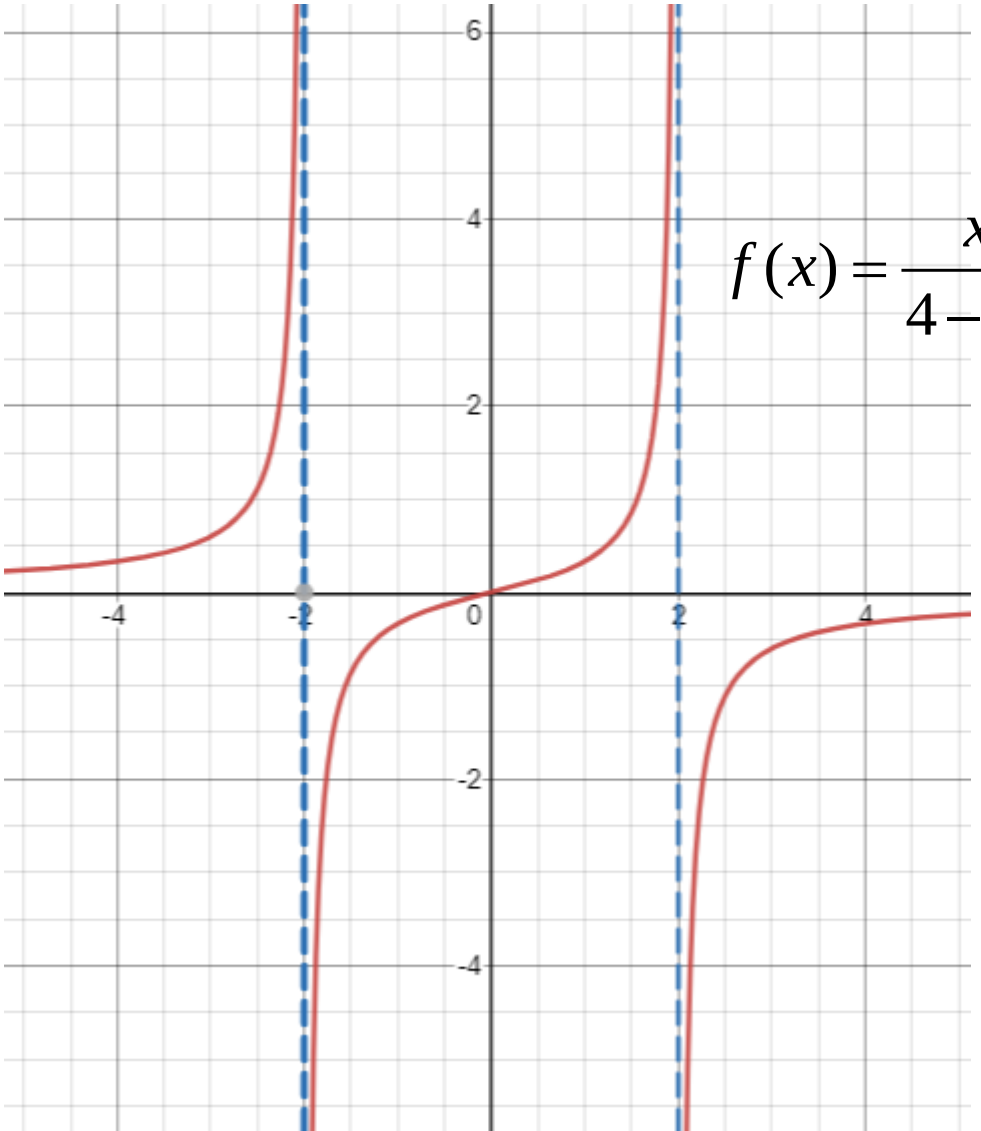
$$\lim_{x\to-\infty}\frac{x}{4-x^2}=\lim_{x\to-\infty}\frac{\frac{x}{x^2}}{\frac{4}{x^2}-\frac{x^2}{x^2}}=\lim_{x\to-\infty}\frac{\frac{1}{x}}{\frac{4}{x^2}-1}=\frac{0}{0-1}=0$$

and

$$\lim_{x\to+\infty}\frac{x}{4-x^2}=\lim_{x\to-\infty}\frac{\frac{x}{x^2}}{\frac{4}{x^2}-\frac{x^2}{x^2}}=\lim_{x\to-\infty}\frac{\frac{1}{x}}{\frac{4}{x^2}-1}=\frac{0}{0-1}=0$$

Therefore y = 0 is a horizontal asymptote.

Graph



$$f(x) = \frac{x}{4 - x^2}$$

1.2 Continuity

- Continuity at a point
- Continuity on an interval

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Continuity at a Point

The limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at* a.

We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language.

(A continuous process is one that takes place gradually, without interruption or abrupt change.)

1 Definition A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

Continuity

Notice that Definition 1 implicitly requires three things if *f* is continuous at *a*:

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$.

Continuity

- If one or more of the above conditions is not satisfied :
  $f(x)$ **discontinuous** at $x = a$ (**point of discontinuity**)

- If f is continuous for all the points in the open interval (a, b) :
  $f(x)$ **continuous in the interval of** $(\boldsymbol{a}, \boldsymbol{b})$.

- If f is continuous in the interval $(-\infty, +\infty)$:
  $f(x)$ **continuous everywhere**

Continuity

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it. The graph can be drawn without removing your pen from the paper.

Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Example 1

Figure 2 shows the graph of a function *f*. At which numbers is *f* discontinuous? Why?

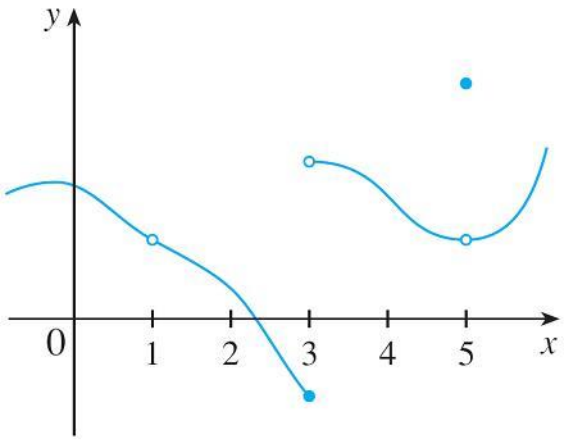


Figure 2

Solution:

1. It looks as if there is a discontinuity when $x = 1$ because the graph has a break there. The official reason that *f* is discontinuous at 1 is that $f(1)$ is not defined.

Example 1 – *Solution*

cont'd

2. The graph also has a break when $x = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

3. What about $x = 5$? Here, $f(5)$ is defined and $\lim_{x \to 5} f(x)$ exists (because the left and right limits are the same).

But

$$\lim_{x \to 5} f(x) \neq f(5)$$

So f is discontinuous at 5.

Example 2

Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Solution:

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

Example 2 – *Solution*

cont'd

(b) Here $f(0) = 1$ is defined but

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2}$$

does not exist. So f is discontinuous at 0.

(c) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2}$$

Example 2 – *Solution*

cont'd

$$= \lim_{x \to 2} (x + 1)$$

= 3 exists.

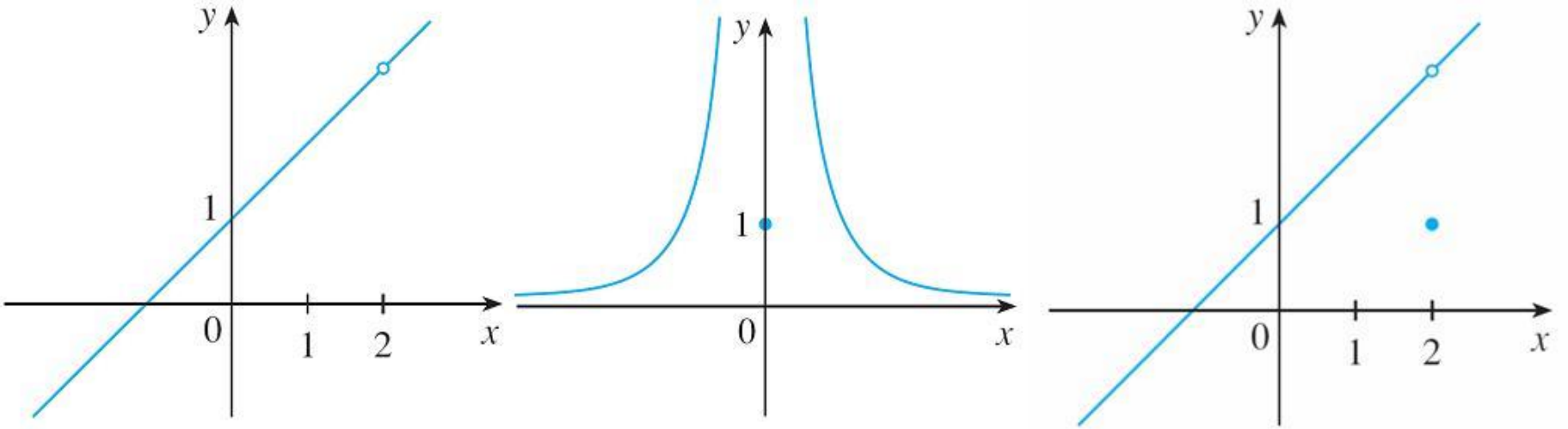
But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so *f* is not continuous at 2.

Continuity

Figure 3 shows the graphs of the functions in Example 2.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Graphs of the functions in Example 2

Figure 3

Example 3

1.Let $f(x) = \frac{x^2-4x-5}{x+1}$. Is $f(x)$ continuous at $x = -1$?

2. Let $f(x) = \begin{cases} x^2 + 1, x < 2 \\ x + 3, x \geq 2 \end{cases}$. Is $f(x)$ continuous at $x = 2$?

Solution Example 3

Example

Given the function

$$f(x)=\begin{cases}\dfrac{3x-6}{2x^3-3x^2-4}, & x\neq 2\\ k, & x=2\end{cases}$$

Find the value of k so that $f(x)$ is continuous at $x = 2$.

Solution:

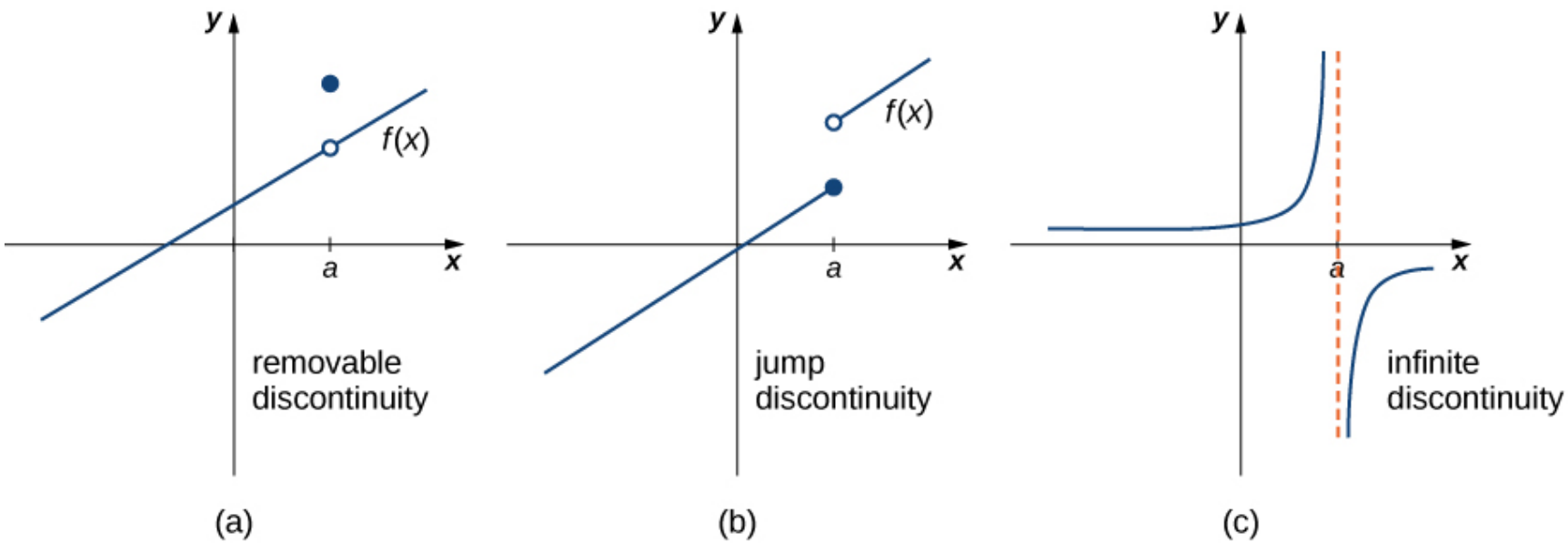
$$\lim_{x\to 2}\frac{3x-6}{2x^3-3x^2-4}=\lim_{x\to 2}\frac{3(x-2)}{(x-2)(2x^2+x+2)}$$

$$=\lim_{x\to 2}\frac{3}{(2x^2+x+2)}$$

$$=\frac{3}{12}=\frac{1}{4}$$

$f(x)$ is continuous of $x = 2$ when $\lim_{x\to 2} f(x) = f(2)$.

As $f(2)=k, \therefore k=\dfrac{1}{4}$



f has a **removable discontinuity** at a if $lim_{x \to a} f(x)$ exists.

f has a **jump discontinuity** at a if $lim_{x \to a^-} f(x)$ and $lim_{x \to a^+} f(x)$ both exist, but $lim_{x \to a^-} f(x) \neq lim_{x \to a^+} f(x)$.

f has an **infinite discontinuity** at a if $lim_{x \to a^-} f(x) = \pm\infty$ or $lim_{x \to a^+} f(x) = \pm\infty$.

Continuity

In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph.

The kind of discontinuity illustrated in parts (a) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.]

The discontinuity in part (c) is called an **infinite discontinuity**. The discontinuities in part (b) are called **jump discontinuities** because the function "jumps" from one value to another.

Practise 1.6

1. Find the value or values of x where the following function discontinuous.

   a) $\frac{x^2-9}{x^2-5x+6}$ b) $\frac{x^3+4x+6}{x^2-6x+8}$

2. Discuss the discontinuity of the function f which is defined by

$$f(x) = \begin{cases} \dfrac{1}{(x-3)(x+2)}, x < 0 \\ \dfrac{1}{(x-2)(x+3)}, x \geq 0 \end{cases}$$

3. The function of f is defined by

$$(x) = \begin{cases} x^2 - 1, x \leq 1 \\ k(x+1), x > 1 \end{cases}$$

   Find the value of k so that the function f is continuous.

Continuity in an Interval

Let f be a function defined in the interval of $[a, b]$.

2 **Definition** A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** b if

$$\lim_{x \to b} f(x) = f(b)$$

3 **Definition** A function f is **continuous on an interval** if it is continuous at the interval of (a,b)

Continuity

The following theorem gives some basic results about continuity.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$
6. f^n

Example

State the interval over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution:

Since $f(x) = \frac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain.

Domain of f(x) = $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$.

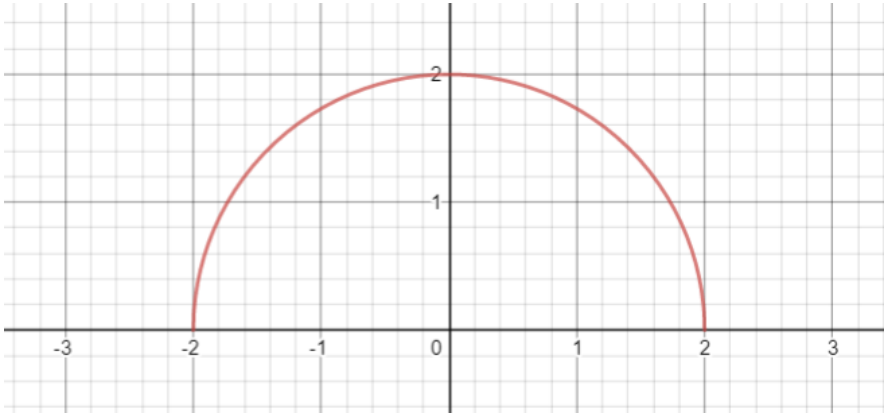
Thus f(x) is continuous over each of the intervals

$$(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$$

Example

If $f(x) = \sqrt{4 - x^2}$, sketch the graph of $y = f(x)$ and prove that f is continuous in the interval $[-2,2]$

Solution:



For $-2 < c < 2$,

1 $\lim_{x \to c} \sqrt{4 - x^2} = \sqrt{4 - c^2} = f(c)$

2 $\lim\limits_{x \to -2^+} \sqrt{4 - x^2} = \sqrt{4 - (-2)^2} = 0 = f(-2)$

3 $\lim\limits_{x \to 2^-} \sqrt{4 - x^2} = \sqrt{4 - (2)^2} = 0 = f(2)$

Hence f is continuous from left of 2 and from the right of -2, therefore the function f is continuous in the interval [-2,2].

Example

Let $h(x) = \begin{cases} x & x < 4 \\ \frac{x}{4} + 2 & 4 \le x \end{cases}$

Is h continuous on the interval $[2,5]$?

(1) $\lim_{x \to 2^+} x = 2$

$f(2) = 2.$

$\lim_{x \to 2^+} x = f(2)$

$\lim_{x \to 5^-} \frac{x}{4} + 2 = 3.25$

$f(5) = 3.25$

$\lim_{x \to 5^-} \frac{x}{4} + 2 = f(5)$

1st condition fulfilled

(2) i) $f(4) = 3$

ii) $\lim_{x \to 4^+} \frac{x}{4} + 2 = 3$

$\lim_{x \to 4^-} x = 4$

$\therefore \lim_{x \to 4}$ not exist

therefore h is not continuous on the interval $(2,5)$.

Practise 1.7

1. Determine the continuity of f(x) at x=a for each of the following

a) $f(x) = \frac{2}{x^2-1}, a = 2$

b) $f(x) = \frac{2x-3}{x-2}, a = 2$

c) $f(x) = \sqrt{5+2x}, a = 2$

2. Determine whether f(x) is continuous at x=2 $f(x) = \begin{cases} \sqrt{x^2+5}, x \leq 2 \\ \frac{x^3+x^2-6}{x}, x > 2 \end{cases}$

3.The function $f(x)$ is defined as $f(x) = \begin{cases} ax^2 - 1, & x \leq 2 \\ bx - 2, & 2 < x < 4 \\ 6, & x \geq 4 \end{cases}$

is continuous in the interval [2,-6] where a and b are constants.

Calculate the value of a and b.

4. Let $g(x) = \begin{cases} x, x < 4 \\ \frac{x}{4} + 2, x \geq 4 \end{cases}$. Is h continuous on the interval (2,5)